Supersymmetric Kadomtsev–Petviashvili hierarchy: ''Ghost'' symmetry structure, reductions, and Darboux–Bäcklund solutions

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This paper studies Manin–Radul supersymmetric Kadomtsev–Petviashvili hierarchy (MR-SKP) in three related aspects: (i) We find an infinite set of additional (''ghost'') symmetry flows spanning the same (anti)commutation algebra as the ordinary MR-SKP flows. (ii) The latter are used to construct consistent reductions $\mathrm{SKP}_{r/2,m/2}$ of the initial unconstrained MR-SKP hierarchy which involves a nontrivial modification for the fermionic flows. (iii) For the simplest constrained MR-SKP hierarchy $\mathrm{SKP}_{\frac{1}{2},\frac{1}{2}}$ we show that the orbit of Darboux–Bäcklund transformations lies on a supersymmetric Toda lattice being a square root of the standard one-dimensional Toda lattice, and also we find explicit Wronskian-ratio solutions for the super-tau function.
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I. INTRODUCTION

Supersymmetric integrable hierarchies of nonlinear evolution (''super-soliton'') equations were originally proposed[1] from purely mathematical motivations, but soon they attracted active interest also in theoretical physics mainly due to their close connections with superstring theory[2] (for related studies of supersymmetric integrable systems of Korteveg–de Vries or nonlinear-Schrödinger type, see Ref. 3).

The scope of the present paper is the supersymmetric Manin–Radul Kadomtsev–Petviashvili (MR-SKP) hierarchy[1] of integrable supersoliton nonlinear equations within the super-pseudo-differential operator formulation (see also Ref. 4; for other formulations see Ref. 5). We study extensions of the MR-SKP hierarchy incorporating additional (anti)commuting ''ghost'' symmetries, as well as reductions of MR-SKP. We use supersymmetric generalization of several basic concepts in the theory of integrable systems which up to now have been most actively pursued in the context of the ordinary (bosonic) KP hierarchy: Baker–Akhiezer wave functions and tau-functions,[6,7] eigenfunctions, and squared eigenfunction potentials (see Refs. 8 and 9, and references therein).

The advantage of constructing an infinite set of (anti)commuting ''ghost'' symmetries in the supersymmetric context (see Sec. IV below) is twofold. On the one hand, it allows us to double the original supersymmetric hierarchy according to the ''duality'' concept, recently introduced in the context of the ordinary KP hierarchy.[10] On the other hand, using the ''ghost'' symmetries we are able to define systematic reductions of the original MR-SKP model to a broad class of constrained supersymmetric KP hierarchies denoted as $\mathrm{SKP}_{r/2,m/2}$ [see Eq. (5.2) below]. These hierarchies posses correct evolution under both even and odd isospectral flows. The latter turns out to be a nontrivial problem since reductions to $\mathrm{SKP}_{r/2,m/2}$ hierarchies are *incompatible* with the original MR-SKP fermionic flows. We provide a solution to this problem by appropriately modifying

MR-SKP fermionic flows while preserving their original (anti)commutation algebra, i.e., preserving the integrability of the constrained $\mathrm{SKP}_{r/2,m/2}$ systems.

The second part of the paper contains a detailed discussion of the simplest constrained MR-SKP hierarchy—$\mathrm{SKP}_{1/2,1/2}$ [Eq. (5.3) below], for which we construct Darboux–Bäcklund (DB) transformations preserving both types (even and odd) of the isospectral flows. This again is achieved thanks to the above-mentioned modification of the original MR-SKP fermionic flows. Further, we study the pertinent DB orbit and discover a new supersymmetric Toda (s-Toda) lattice structure on it. As a consequence of this result we are able to find explicit Wronskian-ratio representation for corresponding super tau function.

Let us mention that several interesting reduced models of the supersymmetric KP hierarchy have been previously constructed in the literature in terms of super-pseudo-differential operators.[11–14] In particular, the supersymmetric version of AKNS hierarchy was found which allows a description in terms of a bosonic[13] as well as fermionic[14] super-Lax operators. The various properties and superspace formulation of these models were worked out, however, their evolution equations involve only even time flows defining them effectively as reductions of the SKP_2 hierarchy,[11] where only even time flows are present by construction.

II. BACKGROUND ON MANIN–RADUL SUPER-KP HIERARCHY

We shall use throughout the super-pseudo-differential calculus[1] with the following notations: ∂ and $\mathcal{D}=\partial/\partial\theta+\theta\partial$ denote operators, whereas the symbols ∂_x and $\mathcal{D}_\theta$ will indicate application of the corresponding operators on superfield functions. As usual, (x,θ) denote superspace coordinates. For any super-pseudo-differential operator $\mathcal{A}=\Sigma_j a_{j/2}\mathcal{D}^j$ the subscripts $(\pm)$ denote its purely differential part $(\mathcal{A}_+=\Sigma_{j\geqslant 0} a_{j/2}\mathcal{D}^j)$ or its purely pseudodifferential part $(\mathcal{A}_- =\Sigma_{j\geqslant 1} a_{-j/2}\mathcal{D}^{-j})$, respectively. For any $\mathcal{A}$ the super-residuum is defined as $\mathrm{Res}\,\mathcal{A}=a_{-1/2}$. The rules of conjugation within the super-pseudo-differential formalism are as follows:[13] $(\mathcal{A}\mathcal{B})^* =(-1)^{|A||B|}\mathcal{B}^*\mathcal{A}^*$ for any two elements with gradings $|A|$ and $|B|$; $(\partial^k)^*=(-1)^k\partial^k$, $(\mathcal{D}^k)^* =(-1)^{k(k+1)/2}\mathcal{D}^k$ and $u^*=u$ for any coefficient superfield.

Finally, in order to avoid confusion we shall also employ the following notations: for any super-(pseudo-)differential operator $\mathcal{A}$ and a superfield function f, the symbol $\mathcal{A}(f)$ will indicate application (action) of $\mathcal{A}$ on f, whereas the symbol $\mathcal{A}f$ will denote just operator product of $\mathcal{A}$ with the zero-order (multiplication) operator f.

MR-SKP hierarchy is defined through the *fermionic* Lax operator $\mathcal{L}$:

$$\mathcal{L}=\mathcal{D}+f_0+\sum_{j=1}^{\infty} b_j\partial^{-j}\mathcal{D}+\sum_{j=1}^{\infty} f_j\partial^{-j} \tag{2.1}$$

expressed in terms of a *bosonic* ''dressing'' operator $\mathcal{W}$:

$$\mathcal{L}=\mathcal{W}\mathcal{D}\mathcal{W}^{-1},\quad \mathcal{W}=1+\sum_{j=1}^{\infty}\alpha_j\partial^{-j}\mathcal{D}+\sum_{j=1}^{\infty}\beta_j\partial^{-j}, \tag{2.2}$$

where b_j,β_j are bosonic superfield functions whereas f_j,α_j are fermionic ones and where

$$f_0=2\alpha_1,\quad b_1=-\mathcal{D}_\theta\alpha_1,\quad f_1=2\alpha_2-\alpha_1\mathcal{D}_\theta\alpha_1-2\alpha_1\beta_1-\mathcal{D}_\theta\beta_1. \tag{2.3}$$

Remark: The square of MR-SKP Lax operator (2.1) is an even operator of the form

$$\mathcal{L}^2=\partial+\mathcal{D}_\theta b_1\partial^{-1}\mathcal{D}+(2b_2+b_1^2+\mathcal{D}_\theta f_1+b_1\mathcal{D}_\theta f_0)\partial^{-1}+\cdots. \tag{2.4}$$

Note that the zero-order term in $\mathcal{L}^2$ vanishes $\mathcal{D}_\theta f_0+2b_1=0$ due to (2.3).
The Lax evolution equations for MR-SKP read[1]

$$\frac{\partial}{\partial t_l}\mathcal{L} = -[\mathcal{L}_-^{2l},\mathcal{L}] = [\mathcal{L}_+^{2l},\mathcal{L}], \tag{2.5}$$

$$D_n\mathcal{L} = -\{\mathcal{L}_-^{2n-1},\mathcal{L}\} = \{\mathcal{L}_+^{2n-1},\mathcal{L}\} - 2\mathcal{L}^{2n}, \tag{2.6}$$

$$\frac{\partial}{\partial t_l}\mathcal{W} = -(\mathcal{W}\partial^l\mathcal{W}^{-1})_-\mathcal{W}, \quad D_n\mathcal{W} = -(\mathcal{W}\mathcal{D}^{2n-1}\mathcal{W}^{-1})_-\mathcal{W}, \tag{2.7}$$

with the short-hand notations

$$D_n = \frac{\partial}{\partial\theta_n} - \sum_{k=1}^{\infty}\theta_k\frac{\partial}{\partial t_{n+k-1}}, \quad \{D_k,D_l\} = -2\frac{\partial}{\partial t_{k+l-1}}, \tag{2.8}$$

$$(t,\theta)\equiv(t_1\equiv x,t_2,\ldots;\theta,\theta_1,\theta_2,\ldots). \tag{2.9}$$

Accordingly, the super-Zakharov–Shabat (super-ZS) equations take the following form:

$$\frac{\partial}{\partial t_k}\mathcal{L}_+^{2l} - \frac{\partial}{\partial t_l}\mathcal{L}_+^{2k} - [\mathcal{L}_+^{2k},\mathcal{L}_+^{2l}] = 0, \quad \frac{\partial}{\partial t_k}\mathcal{L}_+^{2l-1} - D_l\mathcal{L}_+^{2k} - [\mathcal{L}_+^{2k},\mathcal{L}_+^{2l-1}] = 0, \tag{2.10}$$

$$D_k\mathcal{L}_+^{2l-1} + D_l\mathcal{L}_+^{2k-1} - \{\mathcal{L}_+^{2k-1},\mathcal{L}_+^{2l-1}\} + 2\mathcal{L}_+^{2(k+l-1)} = 0. \tag{2.11}$$

Remark: Let us stress that, unlike the possibility to identify $t_1\equiv x$ [since the zero-order term in $\mathcal{L}^2$ (2.4) vanishes], we *cannot* identify $\theta_1\equiv\theta$. Therefore, there is a nontrivial ''evolution'' already with respect to the lowest fermionic flow D_1 (which cannot in general be identified with $\mathcal{D}$).

The super-Baker–Akhiezer (super-BA) and the adjoint super-BA wave functions are defined as

$$\psi_{\mathrm{BA}}(t,\theta;\lambda,\eta) = \mathcal{W}(\psi_{\mathrm{BA}}^{(0)}(t,\theta;\lambda,\eta)), \quad \psi_{\mathrm{BA}}^{*}(t,\theta;\lambda,\eta) = \mathcal{W}^{*-1}(\psi_{\mathrm{BA}}^{*(0)}(t,\theta;\lambda,\eta)) \tag{2.12}$$

(with η being a fermionic ''spectral'' parameter), in terms of the ''free'' super-BA functions

$$\psi_{\mathrm{BA}}^{(0)}(t,\theta;\lambda,\eta) \equiv e^{\xi(t,\theta;\lambda,\eta)}, \quad \psi_{\mathrm{BA}}^{*(0)}(t,\theta;\lambda,\eta) \equiv e^{-\xi(t,\theta;\lambda,\eta)}, \tag{2.13}$$

$$\xi(t,\theta;\lambda,\eta) = \sum_{l=1}^{\infty}\lambda^l t_l + \eta\theta + (\eta-\lambda\theta)\sum_{n=1}^{\infty}\lambda^{n-1}\theta_n \tag{2.14}$$

for which it holds

$$\frac{\partial}{\partial t_k}\psi_{\mathrm{BA}}^{(0)} = \partial_x^k\psi_{\mathrm{BA}}^{(0)}, \quad D_n\psi_{\mathrm{BA}}^{(0)} = \mathcal{D}_\theta^{2n-1}\psi_{\mathrm{BA}}^{(0)} = \partial_x^{n-1}\mathcal{D}_\theta\psi_{\mathrm{BA}}^{(0)}. \tag{2.15}$$

Accordingly, (adjoint) super-BA wave functions satisfy

$$(\mathcal{L}^2)^{(*}\psi_{\mathrm{BA}}^{(*)} = \pm\lambda\psi_{\mathrm{BA}}^{(*)}, \quad \frac{\partial}{\partial t_l}\psi_{\mathrm{BA}}^{(*)} = \pm(\mathcal{L}^{2l})_+^{(*)}(\psi_{\mathrm{BA}}^{(*)}), \quad D_n\psi_{\mathrm{BA}}^{(*)} = \pm(\mathcal{L}^{2n-1})_+^{(*)}(\psi_{\mathrm{BA}}^{(*)}). \tag{2.16}$$

Correspondingly, the defining equations for arbitrary (adjoint-) super-eigenfunctions (sEFs) are

$$\frac{\partial}{\partial t_l}\Phi = \mathcal{L}_+^{2l}(\Phi), \quad D_n\Phi = \mathcal{L}_+^{2n-1}(\Phi), \quad \frac{\partial}{\partial t_l}\Psi = -(\mathcal{L}^{2l})_+^{*}(\Psi), \quad D_n\Psi = -(\mathcal{L}^{2n-1})_+^{*}(\Psi) \tag{2.17}$$

with supersymmetric ''spectral'' representations (cf. Ref. 9)

$$\Phi(t,\theta)=\int d\lambda\, d\eta\, \varphi(\lambda,\eta)\psi_{\mathrm{BA}}(t,\theta;\lambda,\eta), \quad \Psi(t,\theta)=\int d\lambda\, d\eta\, \varphi^{*}(\lambda,\eta)\psi^{*}_{\mathrm{BA}}(t,\theta;\lambda,\eta). \tag{2.18}$$

For later use let us write down the explicit expression for the ''free'' sEF $\Phi^{(0)}$ of the ''free'' $\mathcal{L}^{(0)}=\mathcal{D}$. Namely, taking into account (2.13)–(2.15) and (2.18) we get (for definiteness, consider bosonic $\Phi^{(0)}$)

$$\frac{\partial}{\partial t_k}\Phi^{(0)}=\partial_x^k\Phi^{(0)}, \quad D_n\Phi^{(0)}=\mathcal{D}_\theta^{2n-1}\Phi^{(0)}, \tag{2.19}$$

$$\begin{aligned}\Phi^{(0)}(t,\theta)&=\int d\lambda\, d\eta\, \varphi^{(0)}(\lambda,\eta)e^{\xi(t,\theta;\lambda,\eta)}\\ &=\int d\lambda\left[\left(1-\theta\sum_{n\geqslant 1}\lambda^n\theta_n\right)\varphi_B(\lambda)+\left(\theta+\sum_{n\geqslant 1}\lambda^{n-1}\theta_n\right)\varphi_F(\lambda)\right]\exp\left(\sum_{l\geqslant 1}\lambda^l t_l\right),\end{aligned} \tag{2.20}$$

where $\varphi^{(0)}(\lambda,\eta)=\varphi_F(\lambda)+\eta\varphi_B(\lambda)$ is arbitrary ''spectral'' density.

The super-tau-function $\tau(t,\theta)$ is related with the super-residues of powers of the super-Lax operator (2.1) as follows:

$$\mathrm{Res}\,\mathcal{L}^{2k}=\frac{\partial}{\partial t_k}\mathcal{D}_\theta \ln\tau, \quad \mathrm{Res}\,\mathcal{L}^{2k-1}=D_k\mathcal{D}_\theta\ln\tau. \tag{2.21}$$

Equation (2.21) follows from the identities

$$\begin{gathered}\frac{\partial}{\partial t_l}\mathrm{Res}\,\mathcal{L}^{2k}=\frac{\partial}{\partial t_k}\mathrm{Res}\,\mathcal{L}^{2l}, \quad \frac{\partial}{\partial t_l}\mathrm{Res}\,\mathcal{L}^{2k-1}=D_k\,\mathrm{Res}\,\mathcal{L}^{2l},\\ D_l\,\mathrm{Res}\,\mathcal{L}^{2k-1}+D_k\,\mathrm{Res}\,\mathcal{L}^{2l-1}+2\,\mathrm{Res}\,\mathcal{L}^{2(k+l-1)}=0,\end{gathered} \tag{2.22}$$

which in turn are easily derived from Eqs. (2.5) to (2.6). In particular, for the coefficients of $\mathcal{L}$ and $\mathcal{W}$ we have

$$b_1=\frac{\partial}{\partial t_1}\ln\tau\equiv\partial_x\ln\tau, \quad \alpha_1=D_1\ln\tau. \tag{2.23}$$

In what follows we shall encounter objects of the form $\mathcal{D}_\theta^{-1}(\Phi\Psi)=\mathcal{D}_\theta\partial_x^{-1}(\Phi\Psi)$ where Φ,Ψ is a pair of sEF and adjoint-sEF. Similarly to the purely bosonic case[15] one can show that application of the inverse derivative on such products is well-defined [up to an overall (t,θ)-independent constant]. Namely, there exists a unique superfield function—supersymmetric ''squared eigenfunction potential'' (super-SEP) $S(\Phi,\Psi)$ such that: $\mathcal{D}_\theta S(\Phi,\Psi)=\Phi\Psi$. More precisely the super-SEP satisfies the relations

$$\frac{\partial}{\partial t_k}S(\Phi,\Psi)=\mathrm{Res}(\mathcal{D}^{-1}\Psi\mathcal{L}^{2k}\Phi\mathcal{D}^{-1}), \quad D_kS(\Phi,\Psi)=\mathrm{Res}(\mathcal{D}^{-1}\Psi\mathcal{L}^{2n-1}\Phi\mathcal{D}^{-1}) \tag{2.24}$$

whose consistency follows from the super-ZS Eqs. (2.10) and (2.11). In particular, Eq. (2.24) for $k=1$ and $n=1$ read

$$\partial_x S(\Phi,\Psi)=\mathrm{Res}(\mathcal{D}^{-1}\Psi\mathcal{L}^2\Phi\mathcal{D}^{-1})=\mathcal{D}_\theta(\Phi\Psi),\quad D_1 S(\Phi,\Psi)=\mathrm{Res}(\mathcal{D}^{-1}\Psi\mathcal{L}\Phi\mathcal{D}^{-1})=\Phi\Psi. \tag{2.25}$$

III. ISSUE OF DARBOUX–BÄCKLUND TRANSFORMATIONS IN MR-SKP HIERARCHY

Consider the ''gauge'' transformation of $\mathcal{L}$ (2.1) of the form

$$\tilde{\mathcal{L}}=\mathcal{T}\mathcal{L}\mathcal{T}^{-1},\quad \mathcal{T}=\chi\mathcal{D}\chi^{-1}, \tag{3.1}$$

which parallels the familiar DB transformation in the purely bosonic case.[15,16] Requiring the transformed Lax operator $\tilde{\mathcal{L}}$ to obey MR-SKP evolution equation of the same form (2.5)–(2.6) as $\mathcal{L}$ implies that $\mathcal{T}$ must satisfy

$$\frac{\partial}{\partial t_l}\mathcal{T}\mathcal{T}^{-1}+(\mathcal{T}\mathcal{L}_+^{2l}\mathcal{T}^{-1})_-=0,\quad D_n\mathcal{T}\mathcal{T}^{-1}-(\mathcal{T}\mathcal{L}_+^{2n-1}\mathcal{T}^{-1})_-=-2(\tilde{\mathcal{L}}^{2n-1})_-. \tag{3.2}$$

The first Eq. (3.2) is exactly analogous to the purely bosonic case and implies that χ must be a sEF (2.17) of $\mathcal{L}$ with respect to the even MR-SKP flows. However, there is a problem with the second Eq. (3.2). Namely, for the general (unconstrained) MR-SKP hierarchy it does not have solutions for χ. In particular, if χ would be a sEF also with respect to fermionic flows [cf. the second Eq. (2.17)], then the left-hand side of second Eq. (3.2) would become zero whereupon we would get the contradictory relation: $(\tilde{\mathcal{L}}^{2n-1})_-=0$.

Thus, we conclude that the DB transformations of the general MR-SKP hierarchy preserve only the bosonic flow equations. In what follows we shall look for consistent solutions of (3.2) in the framework of *constrained* MR-SKP systems which will be achieved thanks to a nontrivial modification of the fermionic MR-SKP flows preserving their anticommutation algebra (2.8).

There is a further essential distinction of DB transformations for MR-SKP hierarchy and its purely bosonic counterpart. Calculating the super-residues of the powers of the DB-transformed Lax operator we obtain

$$\mathrm{Res}\,\tilde{\mathcal{L}}^s=\mathcal{D}_\theta(\chi^{-1}\mathcal{L}_+^s(\chi))+(-1)^{s+1}\,\mathrm{Res}\,\mathcal{L}^s. \tag{3.3}$$

Note the crucial sign factor in front of the second term on the right-hand side of Eq. (3.3). Together with the first Eq. (2.21) it implies for the DB-transformed super-τ function

$$\tilde{\tau}=\chi\tau^{-1} \tag{3.4}$$

in contrast with the bosonic case (where we have $\tilde{\tau}=\chi\tau$).

IV. SUPER-''GHOST'' SYMMETRIES OF MR-SKP HIERARCHY

Consider an infinite set $\{\Phi_{j/2},\Psi_{j/2}\}_{j=0}^\infty$ of pairs of (adjoint-)sEFs of $\mathcal{L}$ where those with integer indices are bosonic, whereas those with half-integer indices are fermionic. Next, let us introduce the following infinite set of super-pseudo-differential operators:

$$\mathcal{M}_{s/2}=\sum_{k=0}^{s-1}\Phi_{(s-1-k)/2}\mathcal{D}^{-1}\Psi_{k/2},\quad s=1,2,..., \tag{4.1}$$

which generate an infinite set of flows $\bar{\partial}_{s/2}(\bar{\partial}_{n-1/2}\equiv\bar{D}_n,\bar{\partial}_k\equiv\partial/\partial\bar{t}_k)$:

$$\bar{\partial}_{s/2}\mathcal{W}=\mathcal{M}_{s/2}\mathcal{W},\quad \bar{D}_n\mathcal{L}=\{\mathcal{M}_{n-1/2},\mathcal{L}\},\quad \frac{\partial}{\partial\bar{t}_k}\mathcal{L}=[\mathcal{M}_k,\mathcal{L}]. \tag{4.2}$$

On (adjoint-)sEFs entering $\mathcal{M}_{s/2}$ we allow a *nonhomogeneous* action of the superflows (4.2) which parallels the construction of generalized ''ghost'' symmetry flows in the bosonic case[10] (nonhomogeneous terms are absent in the traditional approach to ''ghost'' symmetry flows[17]):

$$\bar{\partial}_{s/2}\Phi_{l/2}=\mathcal{M}_{s/2}(\Phi_{l/2})-\Phi_{s+l/2}, \quad \bar{\partial}_{s/2}\Psi_{l/2}=-\mathcal{M}^{*}_{s/2}(\Psi_{l/2})+(-1)^{sl}\Psi_{s+l/2}, \tag{4.3}$$

$$\bar{\partial}_{s/2}F^{(*}=\pm\mathcal{M}^{(*)}_{s/2}(F^{(*}), \tag{4.4}$$

where $F^{(*)}$ is a generic (adjoint-)sEF not belonging to the set $\{\Phi_{j/2},\Psi_{j/2}\}$.

Using (4.3) we arrive at the following:

Proposition 1: The infinite set of superflows $\bar{\partial}_{s/2}$ (4.1) (anti)commute both with the ordinary superflows of MR-SKP (2.5)–(2.6) as well as among themselves:

$$\left[\frac{\partial}{\partial \bar{t}_s},\frac{\partial}{\partial t_l}\right]=\left[\frac{\partial}{\partial \bar{t}_s},D_n\right]=0, \quad \left[\bar{D}_s,\frac{\partial}{\partial t_l}\right]=\{\bar{D}_s,D_n\}=0, \tag{4.5}$$

$$\left[\frac{\partial}{\partial \bar{t}_s},\frac{\partial}{\partial \bar{t}_k}\right]=\left[\frac{\partial}{\partial \bar{t}_s},\bar{D}_n\right]=0, \quad \{\bar{D}_i,\bar{D}_j\}=-2\frac{\partial}{\partial \bar{t}_{i+j-1}} \tag{4.6}$$

meaning that $\mathcal{M}_{s/2}$ obey the following equations:

$$\frac{\partial}{\partial t_k}\mathcal{M}_{s/2}=[\mathcal{L}^{2k}_{+},\mathcal{M}_{s/2}]_{-}, \quad D_n\mathcal{M}_k=[\mathcal{L}^{2n-1}_{+},\mathcal{M}_k]_{-}, \quad D_n\mathcal{M}_{k-1/2}=\{\mathcal{L}^{2n-1}_{+},\mathcal{M}_{k-1/2}\}_{-}, \tag{4.7}$$

$$\frac{\partial}{\partial \bar{t}_k}\mathcal{M}_l-\frac{\partial}{\partial \bar{t}_l}\mathcal{M}_k-[\mathcal{M}_k,\mathcal{M}_l]=0, \quad \frac{\partial}{\partial \bar{t}_k}\mathcal{M}_{l-1/2}-\bar{D}_l\mathcal{M}_k-[\mathcal{M}_k,\mathcal{M}_{l-1/2}]=0, \tag{4.8}$$

$$\bar{D}_k\mathcal{M}_{l-1/2}+\bar{D}_l\mathcal{M}_{k-1/2}-\{\mathcal{M}_{k-1/2},\mathcal{M}_{l-1/2}\}=-2\mathcal{M}_{k+l-1}. \tag{4.9}$$

In checking Eqs. (4.7)–(4.9) we make use of several useful identities for super-pseudo-differential operators:

$$[\mathcal{B}_b,\Phi_{s/2}\mathcal{D}^{-1}\Psi_{k/2}]_{-}=\mathcal{B}_b(\Phi_{s/2})\mathcal{D}^{-1}\Psi_{k/2}-\Phi_{s/2}\mathcal{D}^{-1}\mathcal{B}^{*}_b(\Psi_{k/2}), \tag{4.10}$$

$$[\mathcal{B}_f,\Phi_{s/2}\mathcal{D}^{-1}\Psi_{k/2}]^{(\pm)}_{-}=\mathcal{B}_f(\Phi_{s/2})\mathcal{D}^{-1}\Psi_{k/2}+(-1)^{s}\Phi_{s/2}\mathcal{D}^{-1}\mathcal{B}^{*}_f(\Psi_{k/2}), \tag{4.11}$$

$$(\Phi_{s/2}\mathcal{D}^{-1}\Psi_{k/2})(\Phi_{j/2}\mathcal{D}^{-1}\Psi_{l/2})=\mathcal{X}_{(s,k)}(\Phi_{j/2})\mathcal{D}^{-1}\Psi_{l/2}+(-1)^{k(l+j+1)}\Phi_{s/2}\mathcal{D}^{-1}\mathcal{X}^{*}_{(j,l)}(\Psi_{k/2}), \tag{4.12}$$

$$(\Phi_{j/2}\mathcal{D}^{-1}\Psi_{l/2})^{*}=(-1)^{lj+j+l}\Psi_{l/2}\mathcal{D}^{-1}\Phi_{j/2}, \quad \mathcal{X}_{(s,k)}(\Phi)\equiv\Phi_{s/2}\mathcal{D}^{-1}_{\theta}(\Psi_{k/2}\Phi), \tag{4.13}$$

where $\mathcal{B}_b,\mathcal{B}_f$ indicate arbitrary bosonic/fermionic purely differential super-operators, and $[\cdot,\cdot]^{(\pm)}$ denotes commutator or anticommutator whenever the second element is bosonic/fermionic.

V. CONSTRAINED MR-SKP HIERARCHIES

The super-''ghost''-symmetry flows and the corresponding generating operators $\mathcal{M}_{s/2}$ (4.1) and (4.2) can be used to construct reductions of the full (unconstrained) MR-SKP hierarchy.

Namely, since according to Proposition 1 the super-''ghost'' flows obey the same algebra (4.6) as the original MR-SKP flows, we can identify an infinite subset of the latter with a corresponding infinite subset of the former:

$$\partial_{\ell r/2} = -\bar{\partial}_{\ell m/2}, \quad \ell = 1,2,\ldots, \quad \partial_k \equiv \frac{\partial}{\partial t_k}, \quad \partial_{k-1/2} \equiv D_k, \quad \bar{\partial}_k \equiv \frac{\partial}{\partial \bar{t}_k}, \quad \bar{\partial}_{k-1/2} \equiv \bar{D}_k, \tag{5.1}$$

where (r,m) are some fixed positive integers of equal parity, and retain only these flows as Lax evolution flows (this is a supersymmetric extension of the usual reduction procedure in the purely bosonic case[18]). Equation (5.1) implies the identification $(\mathcal{L}^{r\ell})_- = \mathcal{M}_{\ell m/2}$ for any ℓ and, therefore, the corresponding reduced MR-SKP hierarchy denoted as $\mathrm{SKP}_{r/2,m/2}$ is described by the following constrained super-Lax operator:

$$\mathcal{L}_{(r/2,m/2)} = \mathcal{L}_+^r + \sum_{j=0}^{m-1} \Phi_{m-1-j/2} \mathcal{D}^{-1} \Psi_{j/2}. \tag{5.2}$$

The two simplest constrained MR-SKP Lax operators read

$$\mathcal{L}_{(1/2,1/2)} \equiv \mathcal{L} = \mathcal{D} + f_0 + \Phi_0 \mathcal{D}^{-1} \Psi_0, \tag{5.3}$$

$$\mathcal{L}_{(1,1)} = \partial + \Phi_0 \mathcal{D}^{-1} \Psi_{1/2} + \Phi_{1/2} \mathcal{D}^{-1} \Psi_0, \tag{5.4}$$

where Φ_0, Ψ_0 and $\Phi_{1/2}$, $\Psi_{1/2}$ are pairs of bosonic and fermionic (adjoint-)sEFs with respect to the bosonic flows (about the fermionic flows, see below).

In what follows we shall consider in some detail the simplest constrained $\mathrm{SKP}_{1/2,1/2}$ hierarchy (5.3), and henceforth we shall skip the subscript $(\frac{1}{2}, \frac{1}{2})$ of (5.3) for brevity.

Using identities (4.10)–(4.12) we find the identity for any integer power N (for an analogous formula in the purely bosonic case, see Ref. 19):

$$(\mathcal{L}^N)_- = \sum_{j=0}^{N-1} \mathcal{L}^{N-j-1}(\Phi_0) \mathcal{D}^{-1} \mathcal{L}^{j*}(\Psi_0). \tag{5.5}$$

In particular, for the square of (5.3) we get

$$\mathcal{L}^2 = \partial + \mathcal{L}(\Phi_0) \mathcal{D}^{-1} \Psi_0 + \Phi_0 \mathcal{D}^{-1} \mathcal{L}^*(\Psi_0), \tag{5.6}$$

where again the zero-order term $\mathcal{D}_\theta f_0 + 2\Phi_0\Psi_0 = 0$ as a particular case of (2.3).

The constrained MR-SKP Lax operator (5.3) satisfies consistently the bosonic flow Eq. (2.5). However, we need to make a nontrivial modification of the original fermionic flows (2.6) in order to keep them compatible with the reduction from the general to the constrained MR-SKP hierarchy. Indeed, taking the $(-)$ part of Eq. (2.6) for the constrained $\mathcal{L}$ (5.3) and using identity (4.11) together with (5.5) we obtain

$$(D_n \Phi_0 - \mathcal{L}_+^{2n-1}(\Phi_0)) \mathcal{D}^{-1} \Psi_0 - \Phi_0 \mathcal{D}^{-1} (D_n \Psi_0 + (\mathcal{L}^{2n-1})_+^*(\Psi_0))$$
$$= -2(\mathcal{L}^{2n})_- = -2 \sum_{j=0}^{2n-1} \mathcal{L}^{2n-1-j}(\Phi_0) \mathcal{D}^{-1} \mathcal{L}^{j*}(\Psi_0), \tag{5.7}$$

which leads to apparent contradiction.

In Ref. 8 we solved the problem of incompatibility of the standard Orlov–Schulman additional nonisospectral symmetry flows[20] with the reductions of the full bosonic KP hierarchy by appropriately modifying the original Orlov–Schulman flows. Motivated by this work[8] we arrive at the following important proposition:

Proposition 2: There exists the following consistent modification of MR-SKP flows D_n (2.6) *for constrained* $\mathrm{SKP}_{1/2,1/2}$ *hierarchy:*

$$D_n\mathcal{L} = -\{\mathcal{L}_-^{2n-1} - X^{(2n-1)}, \mathcal{L}\} = \{\mathcal{L}_+^{2n-1}, \mathcal{L}\} + \{X^{(2n-1)}, \mathcal{L}\} - 2\mathcal{L}^{2n}, \tag{5.8}$$

$$X^{(2n-1)} \equiv 2\sum_{l=0}^{n-2} \mathcal{L}^{2(n-l)-3}(\Phi_0)\mathcal{D}^{-1}(\mathcal{L}^{2l+1})^*(\Psi_0), \tag{5.9}$$

$$D_n\Phi_0 = \mathcal{L}_+^{2n-1}(\Phi_0) - 2\mathcal{L}^{2n-1}(\Phi_0) + X^{(2n-1)}(\Phi_0), \tag{5.10}$$

$$D_n\Psi_0 = -(\mathcal{L}^{2n-1})_+^*(\Psi_0) + 2(\mathcal{L}^{2n-1})^*(\Psi_0) - (X^{(2n-1)})^*(\Psi_0). \tag{5.11}$$

The modified D_n *flows obey the same anticommutation algebra (2.8) as in the original unconstrained case.*

In checking the correct anticommutation algebra for D_n (5.8) one has to verify the identities

$$D_kX^{(2l-1)} + D_lX^{(2k-1)} - \{X^{(2k-1)}, X^{(2l-1)}\} - \{X^{(2k-1)}, \mathcal{L}^{2l-1}\}_- - \{X^{(2l-1)}, \mathcal{L}^{2k-1}\}_- = 0, \tag{5.12}$$

which in turn follow from the definition of $X^{(2n-1)}$ (5.9) together with identities (4.10)–(4.13).

Remark: It is straightforward to generalize Proposition 2 for arbitrary constrained $\mathrm{SKP}_{r/2,m/2}$ hierarchy (5.2). Namely, the modified fermionic flows have the same form as in (5.8) where in the expression for $X^{(2n-1)}$ [cf. (5.9)] one has to sum over all pairs of (adjoint-) sEFs entering the purely pseudodifferential part of $\mathcal{L}_{(r/2,m/2)}$ in (5.2).

Let us now consider DB transformations on $\mathcal{L} \equiv \mathcal{L}_{(1/2,1/2)}$ (5.3) preserving its constrained form:

$$\tilde{\mathcal{L}} = \mathcal{T}\mathcal{L}\mathcal{T}^{-1} = \mathcal{D} + \tilde{f}_0 + \tilde{\Phi}_0\mathcal{D}^{-1}\tilde{\Psi}_0, \quad \mathcal{T} = \Phi_0\mathcal{D}\Phi_0^{-1}, \tag{5.13}$$

$$\tilde{f}_0 = -f_0 - 2\mathcal{D}_\theta \ln \Phi_0, \quad \tilde{\Phi}_0 = \mathcal{T}\mathcal{L}(\Phi_0) = \Phi_0\partial_x \ln \Phi_0 + \Phi_0\mathcal{D}_\theta f_0 + \Phi_0^2\Psi_0, \quad \tilde{\Psi}_0 = \Phi_0^{-1}. \tag{5.14}$$

We have the following useful identities for DB-transformed quantities:

$$\tilde{\mathcal{L}}^s(\tilde{\Phi}_0) = \mathcal{T}\mathcal{L}^{s+1}(\Phi_0),$$

$$(\tilde{\mathcal{L}}^{s+1})^*(\tilde{\Psi}_0) = (-1)^{s+1}\mathcal{T}^{*-1}\mathcal{L}^{s*}(\Psi_0) = (-1)^s\Phi_0^{-1}\mathcal{D}_\theta^{-1}(\Phi_0\mathcal{L}^{s*}(\Psi_0)). \tag{5.15}$$

There is a further crucial property of the modified D_n flows (5.8)–(5.9):

Proposition 3: The conditions for preserving the fermionic flow Eqs. (5.8)–(5.9) by the Darboux–Bäcklund transformations on $\mathcal{L} \equiv \mathcal{L}_{1/2,1/2}$ *(5.3) [cf. second Eq. (3.2)]:*

$$D_n\mathcal{T}\mathcal{T}^{-1} - (\mathcal{T}\mathcal{L}_+^{2n-1}\mathcal{T}^{-1})_- = -2(\tilde{\mathcal{L}}^{2n-1})_- + \tilde{X}^{(2n-1)} + \mathcal{T}X^{(2n-1)}\mathcal{T}^{-1}, \tag{5.16}$$

where $\mathcal{T} = \Phi_0\mathcal{D}\Phi_0^{-1}$ *and the "tilde" refers to DB-transformed objects, are now satisfied.* The proof of (5.16) proceeds by using the modified D_n flow definitions (5.9)–(5.11) together with identities (4.10)–(4.13) and (5.15).

VI. THE DARBOUX–BÄCKLUND ORBIT OF THE CONSTRAINED MR-SKP HIERARCHY

The recursive expression for the chain of the DB-transformations (5.13)–(5.14) of the constrained $\mathrm{SKP}_{1/2,1/2}$ hierarchy, starting from the "free" initial $\mathcal{L}_0 = \mathcal{D}$, reads (the subscript k indicating the step of DB iteration)

$$\mathcal{L}_{k+1} = \mathcal{T}_k\mathcal{L}_k\mathcal{T}_k^{-1} = \mathcal{D} + f_{k+1} + \Phi_{k+1}\mathcal{D}^{-1}\Psi_{k+1}, \quad \mathcal{T}_k = \Phi_k\mathcal{D}\Phi_k^{-1}, \tag{6.1}$$

$$\mathcal{L}_1=\mathcal{T}_0\mathcal{D}\mathcal{T}_0^{-1}=\mathcal{D}-2\mathcal{D}_\theta \ln\Phi_0+\Phi_0(\partial_x \ln\Phi_0)\mathcal{D}^{-1}\Phi_0^{-1}, \tag{6.2}$$

where

$$f_{k+1}=-2\mathcal{D}_\theta \ln\Phi_k-f_k, \quad \Psi_{k+1}=\Phi_k^{-1}, \tag{6.3}$$

$$\Phi_{k+1}=\Phi_k\partial_x \ln\Phi_k+\Phi_k\mathcal{D}_\theta f_k+\Phi_k^2\Psi_k \tag{6.4}$$

and where Φ_0 is a sEF of the initial ''free'' $\mathcal{L}_0=\mathcal{D}$ satisfying the ''free'' version of Eq. (5.10) (no $X^{(2n-1)}$ term). Therefore, its explicit expression is given by Eq. (2.20) with substituting $\theta_n\to -\theta_n$. Further we have

$$\Phi_1=\partial_x\Phi_0, \quad \Psi_1=\Phi_0^{-1}, \quad f_1=-2\mathcal{D}_\theta \ln\Phi_0. \tag{6.5}$$

Note, that from (6.3) to (6.4) we find

$$2\Phi_{k+1}\Psi_{k+1}+\mathcal{D}_\theta f_{k+1}=2\Phi_k\Psi_k+\mathcal{D}_\theta f_k=\cdots=0, \tag{6.6}$$

which is consistent with the absence of a zero-order term in the square of $\mathcal{L}_k$ in (6.1).

Equation (6.3) can easily be rewritten as follows:

$$f_{k+1}=-2\mathcal{D}_\theta\sum_{i=0}^{k}(-1)^{k-i}\ln\Phi_i. \tag{6.7}$$

Recalling identity (6.6) we can alternatively rewrite Eq. (6.4) as

$$\Phi_{k+1}=-\tfrac{1}{2}\Phi_k\mathcal{D}_\theta f_{k+1}=\Phi_k\partial_x \ln\Phi_k-\Phi_k^2\Psi_k \tag{6.8}$$

from which we obtain

$$\Phi_{k+1}=\Phi_k\sum_{i=0}^{k}(-1)^{k-i}\partial_x \ln\Phi_i. \tag{6.9}$$

After making the standard substitution $\Phi_k=e^{\varphi_k}$, we find from the second equation in (6.8) a *new* super-Toda (**s**-Toda) lattice equation:

$$\partial_x\varphi_k=e^{\varphi_{k+1}-\varphi_k}+e^{\varphi_k-\varphi_{k-1}}. \tag{6.10}$$

Note, that by acting on (6.10) with ∂_x we get

$$\partial_x^2\varphi_k=e^{\varphi_{k+2}-\varphi_k}-e^{\varphi_k-\varphi_{k-2}}, \tag{6.11}$$

which has the form of the ordinary one-dimensional Toda lattice equation but with a *doubled* lattice spacing and, of course, the Toda variables $\varphi_k=\varphi_k(x,t_2,\ldots;\theta,\theta_1,\ldots)$ are now superfields. Equation (6.10) can also be rewritten as

$$e^{\varphi_{k+1}-\varphi_k}=\sum_{i=0}^{k}(-1)^{k-i}\partial_x\varphi_i \tag{6.12}$$

or

$$\varphi_{k+1}=\varphi_k+\ln\left(\sum_{i=0}^{k}(-1)^{k-i}\partial_x\varphi_i\right). \tag{6.13}$$

We now discuss the Wronskian representation for the sEFs Φ_k. The **s**-Toda lattice (6.10) can apparently be thought of as the square root of the standard Toda lattice. We can use this idea to proceed without any technical calculations. According to the construction given in Ref. 21 the EFs Φ_{2n} associated with even lattice points can be given the usual Wronskian expressions with the starting ''point'' Φ_0. For the same reason, EFs Φ_{2n+1} associated with odd lattice points of the **s**-Toda lattice will have the usual Wronskian expressions with the starting ''point'' $\Phi_1=\partial_x\Phi_0 \equiv\Phi_0^{(1)}$ (6.5).

Generally, for $n=0,1,...$ we find by the above arguments

$$\Phi_{2n}=\frac{W_{n+1}[\Phi_0,\Phi_0^{(1)},\dots,\Phi_0^{(n)}]}{W_n[\Phi_0,\Phi_0^{(1)},\dots,\Phi_0^{(n-1)}]},\quad \Phi_{2n+1}=\frac{W_{n+1}[\Phi_0^{(1)},\Phi_0^{(2)},\dots,\Phi_0^{(n+1)}]}{W_n[\Phi_0^{(1)},\Phi_0^{(2)},\dots,\Phi_0^{(n)}]}, \tag{6.14}$$

where $W_k[f_1,\dots,f_k]\equiv\det\|\partial_x^{i-1}f_j\|$, $i,j=1,...,k$, denotes standard Wronskian determinant [however, with superfield entries in (6.14)] and where $\Phi_0^{(k)}\equiv\partial_x^k\Phi_0$ with Φ_0 as in (2.20) (with $\theta_n\to-\theta_n$).

Using (3.4) and the above Wronskians expressions (6.14) we find by iteration the super-tau functions obtained by $2n$ recursive steps of the DB transformations:

$$\tau^{(2n)}=\frac{\Phi_{2n-1}\Phi_{2n-3}\cdots\Phi_1}{\Phi_{2n-2}\Phi_{2n-4}\cdots\Phi_0}=\frac{W_n[\Phi_0^{(1)},\dots,\Phi_0^{(n)}]}{W_n[\Phi_0,\Phi_0^{(1)},\dots,\Phi_0^{(n-1)}]}, \tag{6.15}$$

$$\tau^{(2n+1)}=\frac{\Phi_{2n}\Phi_{2n-2}\cdots\Phi_0}{\Phi_{2n-1}\Phi_{2n-3}\cdots\Phi_1}=\frac{W_{n+1}[\Phi_0,\Phi_0^{(1)},\dots,\Phi_0^{(n)}]}{W_n[\Phi_0^{(1)},\dots,\Phi_0^{(n)}]}. \tag{6.16}$$

Moreover, since for (5.3) $\partial_x \ln\tau=\Phi_0\Psi_0$, for the k-step DB iteration we have $\partial_x \ln\tau^{(k)}=\Phi_k/\Phi_{k-1}$ by taking into account (3.4). The latter equation together with the relation $\tau^{(k+1)}=\Phi_k/\tau^{(k)}$ true for any DB-step k [cf. (3.4)] yields an alternative super-tau-function form of **s**-Toda lattice:

$$\partial_x \ln\tau^{(k)}(t,\theta)=\frac{\tau^{(k+1)}(t,\theta)}{\tau^{(k-1)}(t,\theta)} \tag{6.17}$$

with the short-hand notation (2.9).

In a subsequent paper we plan to discuss several interesting issues connected with extending the present results: (a) construction of a ''doubled'' MR-SKP hierarchy by providing a super-Lax formulation for the super-''ghost'' symmetry flows [cf. (4.5)–(4.6)]–a supersymmetric extension of the double-KP construction of Ref. 10; (b) general treatment of arbitrary constrained $\mathrm{SKP}_{r/2,m/2}$ hierarchies, including derivation of more general Wronskian-type solutions for the super-tau function and elucidating their Berezinian origin; (c) obtaining consistent formulation of supersymmetric two-dimensional Toda lattice as Darboux–Bäcklund orbit on the ''doubled'' MR-SKP hierarchy (similar to the purely bosonic case[10]) and of supersymmetric analogs of random (multi)matrix models; (d) study of possible connections of super-tau functions, on one hand, and partition functions and joint distribution functions in random matrix models in condensed matter physics (cf. Ref. 22), on the other hand.

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[1] Yu. Manin and A. Radul, Commun. Math. Phys. **98**, 65 (1985).
[2] L. Alvarez-Gaumé, H. Itoyama, J. Mañes, and A. Zadra, Int. J. Mod. Phys. A **7**, 5337 (1992); L. Alvarez-Gaumé, K. Becker, M. Becker, R. Emperan, and J. Mañes, *ibid.* **8**, 2297 (1993); S. Stanciu, Commun. Math. Phys. **165**, 261 (1994), hep-th/9407189; J. M. Figueroa-O'Farrill and S. Stanciu, Phys. Lett. B **316**, 282 (1993).
[3] P. Mathieu, J. Math. Phys. **29**, 2499 (1988); S. Belluci, E. Ivanov, S. Krivonos, and A. Pichugin, Phys. Lett. B **312**, 463

(1993), hep-th/9305078; F. Delduc, E. Ivanov, and S. Krivonos, J. Math. Phys. **37**, 1356 (1996), hep-th/9510033; F. Toppan, Int. J. Mod. Phys. A **11**, 3257 (1996), hep-th/9506133; Q. P. Liu and M. Mañas, solv-int/9711002.
[4] L. Martinez Alonso and E. Medina Reus, J. Math. Phys. **36**, 4898 (1995); A. Ibort, L. Martinez Alonso, and E. Medina Reus, *ibid.* **37**, 6157 (1996).
[5] V. Kac and J. van de Leur, Ann. Inst. Fourier **37**, 99 (1987); V. Kac and E. Medina, Lett. Math. Phys. **37**, 435 (1996); A. LeClair, Nucl. Phys. B **314**, 425 (1989); M. Mulase, J. Diff. Geom. **34**, 651 (1991); J. Rabin, Commun. Math. Phys. **137**, 533 (1991).
[6] L. Dickey, *Soliton Equations and Hamiltonian Systems* (World Scientific, Singapore 1991); Acta Applicandae Math. **47**, 243 (1997).
[7] P. van Moerbeke, in *Lectures on Integrable Systems*, edited by O. Babelon *et al.* (World Scientific, Singapore, 1994).
[8] H. Aratyn, E. Nissimov, and S. Pacheva, Int. J. Mod. Phys. A **12**, 1265 (1997), hep-th/9607234; Phys. Lett. A **228**, 164 (1997), hep-th/9602068.
[9] H. Aratyn, E. Nissimov, and S. Pacheva, Commun. Math. Phys. **193**, 493 (1998), solv-int/9701017.
[10] H. Aratyn, E. Nissimov, and S. Pacheva, Phys. Lett. A **244**, 245 (1998), solv-int/9712012.
[11] J. M. Figueroa-O'Farrill, J. Mas, and E. Ramos, Rev. Mod. Phys. **3**, 479 (1991); W. Oevel and Z. Popowicz, Commun. Math. Phys. **139**, 441 (1991); F. Yu, Nucl. Phys. B **375**, 173 (1992).
[12] Z. Popowicz, J. Phys. A **29**, 1281 (1996), hep-th/9510185; J. C. Brunelli and A. Das, Phys. Lett. B **337**, 303 (1994), hep-th/9406214; L. Bonora, S. Krivonos, and A. Sorin, Nucl. Phys. B **477**, 835 (1996), hep-th/9604165; F. Delduc and L. Gallot, Commun. Math. Phys. **190**, 395 (1997), solv-int/9609008; J.-C. Shaw and M.-H. Tu, solv-int/9712009.
[13] H. Aratyn and C. Rasinariu, Phys. Lett. B **391**, 99 (1997), hep-th/9608107; H. Aratyn, A. Das, and C. Rasinariu, Mod. Phys. Lett. A **12**, 2623 (1997), solv-int/9704119; H. Aratyn, A. Das, C. Rasinariu, and A. H. Zimerman, in *Supersymmetry and Integrable Models*, Lecture Notes in Physics Vol. 502, edited by H. Aratyn *et al.* (Springer-Verlag, Berlin, 1998).
[14] H. Aratyn and A. Das, Mod. Phys. Lett. A **13**, 1185 (1998), solv-int/9710026.
[15] W. Oevel, Physica A **195**, 533 (1993); W. Oevel and W. Schief, Rev. Math. Phys. **6**, 1301 (1994).
[16] L.-L. Chau, J. C. Shaw, and H. C. Yen, Commun. Math. Phys. **149**, 263 (1992).
[17] A. Orlov, in *Nonlinear Processes in Physics: Proceedings of the III Potsdam-V Kiev Workshop*, edited by A. S. Fokas *et al.* (Springer, Berlin, 1993); Y. Cheng, W. Strampp, and B. Zhang, Commun. Math. Phys. **168**, 117 (1995).
[18] W. Oevel, Physica A **195**, 533 (1993); Y. Cheng, W. Strampp, and B. Zhang, Commun. Math. Phys. **168**, 117 (1995).
[19] B. Enriquez, A. Yu. Orlov, and V. N. Rubtsov, Inverse Probl. **12**, 241 (1996), solv-int/9510002.
[20] A. Orlov and E. Schulman, Theor. Math. Phys. **64**, 323 (1985); Lett. Math. Phys. **12**, 171 (1986); A. Orlov, in *Plasma Theory and Nonlinear and Turbulent Processes in Physics* (World Scientific, Singapore, 1988).
[21] H. Aratyn, E. Nissimov, and S. Pacheva, Phys. Lett. A **201**, 293 (1995), hep-th/9501018.
[22] Y. Avishai, Y. Hatsugai, and M. Kamoto, Phys. Rev. B **53**, 8369 (1996).